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ABOUT THE PREDICTION OF TURBULENT PRANDTL AND SCHMIDT NUMBERS FROM MODELED TRANSPORT EQUATIONS

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Abstract—Relationships for the turbulent Prandtl and Schmidt number are presented. They are based on modeled transport equations for the turbulent kinetic energy, for the turbulent heat flux, and the turbulent mass flux. The final results are simple formulae $Pr_t = C+B/Pr$ and $Sc_t = F+E/Sc$ where the constants have to be adjusted from experimental data. This has been done for the constants in the relationship for the turbulent Prandtl number which is in very good agreement with experiments. The agreement can be improved by assuming B to be a function of the Reynolds number. The result is specially interesting for liquid metals, because for low Prandtl numbers the turbulent Prandtl number obviously depends strongly on the molecular Prandtl number. The influence of the Reynolds number, which is increasing for decreasing Prandtl numbers, seems to be weaker.

NOMENCLATURE

c, mass-fraction;
c_p, specific heat at constant pressure;
D, binary diffusion coefficient;
H, half channel width;
j, diffusion flux;
k, turbulent kinetic energy;
l, mixing length;
L, turbulence length scale;
Le_t, turbulent Lewis number;
p, static pressure;
Pe, Péclet number;
Pr, molecular Prandtl number;
Pr_t, turbulent Prandtl number;
q, heat flux;
Re, Reynolds number;
Sc, molecular Schmidt number;
Sc_t, turbulent Schmidt number;
T, temperature;
u,v,w, components of velocity;
x,y, components of space co-ordinate.

Greek symbols

ε, turbulent dissipation;
ε_D, eddy diffusivity for mass;
ε_h, eddy diffusivity for heat;
ε_m, eddy diffusivity for momentum;
η, dynamic viscosity;
λ, thermal conductivity;
ν, kinematic viscosity;
ρ, density;
τ, shear stress.

Subscripts

t, turbulent.

1. INTRODUCTION

THE KNOWLEDGE of the turbulent Prandtl and Schmidt number is the central problem of all theoretical considerations concerning the turbulent heat and mass transfer. Experimental as well as theoretical investigations have shown that the turbulent Prandtl/Schmidt number is neither a constant nor a unique function of the fluid properties. The turbulent Prandtl/Schmidt number depends strongly on the turbulence mechanism and therefore on the flow field. In general it is a function of:

The Reynolds number Re;
The molecular Prandtl or Schmidt number respectively; and
The coordinate normal to the wall.

We shall discuss in detail in Section 3 that the influence of these three variables depends more or less on the molecular Prandtl/Schmidt number itself. For Pr or $Sc \gtrsim 1$ (gases and liquids) the influence of the Reynolds and Prandtl/Schmidt number seems to be weak. For values Pr or $Sc \ll 1$ (liquid metals) on the other hand there seems to be a strong influence of the Reynolds and Prandtl/Schmidt number on the turbulent Prandtl/Schmidt number. In this case the dependence on the wall distance is small compared with the dependence on the Reynolds and Prandtl/Schmidt number. As a consequence of these statements the exact knowledge of the turbulent Prandtl/Schmidt number is much more problematic for liquid metals than for gases and liquids.

The aim of this work is to discuss and to consider the influence of the molecular Prandtl/Schmidt number on the turbulent Prandtl/Schmidt number. Modeling assumptions will lead to a simple algebraic formula $Pr_t(Pr)$ and $Sc_t(Sc)$ which can be used for fully developed duct flow. Further on we shall discuss the question how the influence of the Reynolds number can be taken into account.

2. GOVERNING EQUATIONS

The following considerations are restricted to two-dimensional incompressible thin shear flow layers of a binary nonreacting mixture which are described by the boundary layer equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\frac{\partial \tau}{\partial y} \tag{2.2}$$

$$\rho\left(u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}\right)=-\frac{\partial j}{\partial y} \tag{2.3}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=-\frac{\partial q}{\partial y} \tag{2.4}$$

$$\tau=\eta\frac{\partial u}{\partial y}-\rho\overline{u'v'} \tag{2.5}$$

$$j=-\rho D\frac{\partial c}{\partial y}+\rho\overline{c'v'} \tag{2.6}$$

$$q=-\lambda\frac{\partial T}{\partial y}+\rho c_p\overline{T'v'}. \tag{2.7}$$

The velocity components u and v, the pressure p, the mass fraction c and the temperature T are time mean values; the bar is omitted for simplicity except for the correlation terms. The density ρ, the specific heat c_p as well as the transport coefficients η, D and λ are assumed to be constant.

In analogy to molecular transport coefficients Boussinesq suggested the introduction of eddy diffusivities by the definitions

$$\overline{u'v'}=-\varepsilon_m\frac{\partial u}{\partial y};\quad \overline{c'v'}=-\varepsilon_D\frac{\partial c}{\partial y};\quad \overline{T'v'}=-\varepsilon_h\frac{\partial T}{\partial y}. \tag{2.8}$$

Now the Reynolds shear stress is replaced by the eddy diffusivity ε_m for momentum and Reynolds heat and mass flux are replaced by the eddy diffusivity ε_h for heat and the eddy diffusivity ε_D for mass. Ratios of those diffusivities are called the turbulent Prandtl and Schmidt number:

$$Pr_t=\frac{\varepsilon_m}{\varepsilon_h}=\frac{\overline{u'v'}}{\overline{T'v'}}\cdot\frac{\partial T/\partial y}{\partial u/\partial y};\quad Sc_t=\frac{\varepsilon_m}{\varepsilon_D}=\frac{\overline{u'v'}}{\overline{c'v'}}\cdot\frac{\partial c/\partial y}{\partial u/\partial y}. \tag{2.9}$$

In addition a turbulent Lewis number is defined as the ratio

$$Le_t=\frac{Pr_t}{Sc_t}. \tag{2.10}$$

Closure assumptions are necessary to determine the unknown Reynolds fluxes or the eddy diffusivities respectively. This can be done either:

by local relationships like the mixing length concept of Prandtl (first order closure); or

by differential equations based on modeled transport equations (second order closure).

3. EXPERIMENTAL INVESTIGATIONS

From experiments in smooth tubes and channels it is known that the turbulent Prandtl number is a function of the Reynolds number

$$Re=\frac{2Hu_m}{\nu};\quad u_m=\text{mixed mean velocity}, \tag{3.1}$$

and the Prandtl number

$$Pr=\frac{\eta c_p}{\lambda}, \tag{3.2}$$

as well as the normalized wall distance y/H. $2H$ is the width of the channel or the diameter of the tube. Therefore a relationship,

$$Pr_t=f(Pr, Re, y/H)\text{ or }Sc_t=f(Sc, Re, y/H), \tag{3.3}$$

is expected. Sometimes equation (3.3) is written like

$$Pr_t=f(Pr, Re, \varepsilon_m/\nu)\text{ or }Sc_t=f(Sc, Re, \varepsilon_m/\nu). \tag{3.4}$$

This can be done because the eddy diffusivity ε_m of momentum divided by the viscosity ν is a function of the Reynolds number and the wall distance.

In the following some experimental results based on the book of Eckert and Drake [1] and the thesis of Fuchs [2] are given. Eckert and Drake report measurements for air and mercury in fully developed turbulent tube flow and they write [1, p. 384]: "Experimental difficulties were obviously so great that no satisfactory agreement between the results obtained by various investigators could be obtained". Fuchs [2] describes the experimental difficulties in more detail. They arise from the definition of the turbulent Prandtl number, equation (2.9), itself. The sources of errors are:

the determination of the temperature and gradient from the measured temperature and velocity profiles; and

the determination of the correlation terms. They can be calculated from equations (2.2), (2.4), (2.5), (2.7) putting in the measured temperature and velocity profiles.

In the work of Fuchs the reader will find an extensive discussion about that point.

In spite of the great experimental difficulties some conclusions can be drawn from the available experiments:

$Pr_t < 1$ for $Pr \gtrsim 1$ (gases and liquids);
$Pr_t > 1$ for $Pr \ll 1$ (liquid metals).

The influence of the wall distance seems not to be great, as measurements show [1, 2]. Additionally it is not fully clear at the moment, whether the turbulent Prandtl number decreases or increases from the wall to the center of a tube (or the outer edge of a boundary layer). There seems to be a trend of decreasing turbulent Prandtl number with increasing distance from the wall, see equation (4.11) for air given by Rotta for example.

On the other hand the influence of the molecular Prandtl number is much greater, this influence increases rapidly with decreasing Prandtl number. In

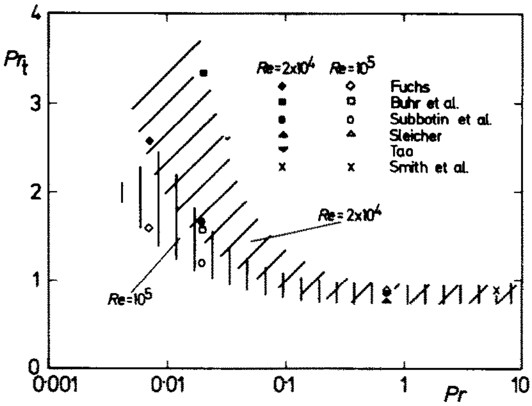

FIG. 1. Experimental results for the turbulent Prandtl number.

Fig. 1 some measurements, which have been reported by Fuchs [2], are shown. These measurements are made in fully developed tube flow by Fuchs [2] for $Pr = 0.007$, Buhr *et al.* [3] and Subbotin *et al.* [4] for $Pr = 0.02$, Sleicher [5] and Tao [6] for $Pr = 0.71$ and Smith *et al.* [7] for $Pr = 6.2$. Two different values of the Reynolds number have been chosen.

The measurements suggest two marked regions for the two Reynolds numbers. Figure 1 shows that:

Pr_t seems to be constant for $Pr > 1$;
the influence of the Prandtl and Reynolds number increases with decreasing Prandtl number; and
Pr_t increases with decreasing Prandtl number and decreasing Reynolds number.

It should be noted that other contributions reported by Reynolds [8] indicate that for small Prandtl numbers the turbulent Prandtl number increases with decreasing Reynolds number as shown in Fig. 1, whereas for great Prandtl numbers there seems to be an opposite effect.

Furthermore it should be repeated that the experimental knowledge in this field is rather poor and that it is risky to draw conclusions from unsecured experimental data. Nevertheless in the following we have some confidence in the information given by Fig. 1.

Experimental data for the turbulent Schmidt number are extremely rare. It is expected that Sc_t and Pr_t would vary in the same way with flow conditions, molecular properties and position in the flow field. The transport equation for the Reynolds shear stress has not precisely the same form as those for the Reynolds mass and heat flux (momentum is a vector whereas mass fraction and energy are scalar quantities), see Section 6. The statement $Le_t = 1$ seems to be generally accepted.

4. FIRST ORDER CLOSURE ASSUMPTIONS FOR THE TURBULENT PRANDTL NUMBER

Already Reynolds suggested that the same mechanism of turbulent exchange causes the transfer of momentum and of heat. This leads to the statement $\varepsilon_m = \varepsilon_h$ or $Pr_t = 1$.

The same result is obtained with the mixing length concept of Prandtl which gives for the Reynolds shear stress and the Reynolds heat flux:

$$\tau_t = -\rho\overline{u'v'} = \rho l^2 \frac{\partial u}{\partial y}\frac{\partial u}{\partial y} \tag{4.1}$$

$$q_t = \rho c_p \overline{T'v'} = \rho c_p l^2 \frac{\partial u}{\partial y}\frac{\partial T}{\partial y}. \tag{4.2}$$

This also leads to the statement $Pr_t = 1$. In contrast to Prandtl, Taylor has used different mixing lengths for the turbulent transfer of momentum and heat and he obtained $Pr_t = 0.5$.

The value 1 is not bad for air flow in ducts and boundary layers, whereas the value 0.5 is approximately valid for air jet flows.

In the past a lot of effort has been made to improve these simple models. Recently Reynolds [9] published a comprehensive review of models which express Pr_t as functions of Pr, of position in the flow field and sometimes of Reynolds number. Remarks about this problem are found in an article by Launder [10], as well as in the work of Fuchs [2], who has made interesting critical comments.

Following the article of Reynolds [9], who examines more than thirty ways of predicting the turbulent Prandtl number, these models can be divided into seven classes. A short recapitulation of Reynolds' work is given in the following. Only a few articles are reported, for details see the original work [9].

The most numerous group of analytical results are three classes based on Prandtl's concept of mixing length:

(a) *Jenkins' [11] analysis and subsequent modifications*

Jenkins modeled the conduction heat loss of a liquid–metal eddy as the transient cooling of a solid sphere during a mean travel time, Jenkins' model was modified by Rohsenow and Cohen [12] who proposed the relationship:

$$Pr_t^{-1} = 416Pr\left|\frac{1}{15} - \frac{6}{\pi^4}\sum_{n=1}^{\infty} n^{-4}\exp\left(\frac{-0.0024\pi^2 n^2}{Pr}\right)\right|. \tag{4.3}$$

The values for liquid metals are about right, but the value of 1.15 for air ($Pr = 0.7$) is too high. Nevertheless in the opinion of White [13] this is the best available theory for the turbulent Prandtl number.

(b) *Deissler's [14] analysis and developments of it*

In contrast to Jenkins, Deissler was only interested in very small molecular Prandtl numbers, he took into account only the heat transfer from the moving element. So Deissler's model differs fundamentally from Jenkins in taking the transfers to and from the sphere to be controlled by diffusive processes external to the moving element, rather than within it.

Deissler's formula reads:

$$Pr_t^{-1} = bPe[1-\exp(-1/(bPe))]. \qquad (4.4)$$

Here Pe is the Péclet number $Pe = RePr$ and $b = 0.000153$ is an empirical constant.

Lykoudis and Touloukian [15] as well as Aoki [16] and others have modified Deissler's analysis in several aspects. Aoki gives

$$Pr_t^{-1} = KRe^{0.45}Pr^{0.2} \times \{1-\exp[-1/(KRe^{0.45}Pr^{0.2})]\}, \qquad (4.5)$$

where $K = 0.014$ is an empirical constant.

(c) *More varied mixing—length models*

The defining feature of this category in contrast to the models of type (a) and (b) is not the use of an immutable mixing length, but the introduction of a discrete element which gains or loses heat and momentum as it moves through the fluid. Typical for this group is the work of Azer and Chao [17]. Although their complete formulae are complicated, they give simpler approximation for pipe flow

$$Pr_t = \frac{1+57f(y/R)/(Re^{0.46}Pr^{0.58})}{1+135f(y/R)/Re^{0.45}} \qquad (4.6)$$

valid for $0.6 \leqslant Pr \leqslant 15$, and

$$Pr_t = \frac{1+380f(y/R)/Pe^{0.58}}{1+135f(y/R)/Re^{0.45}} \qquad (4.7)$$

valid for liquid metals. In both cases, the variation across the pipe is given by

$$f(y/R) = \exp[-(y/R)^{1/4}]. \qquad (4.8)$$

The remaining four groups of models are not based on mixing length concepts; they are more heterogeneous. They range from statistical considerations to simple empiricism. Reynolds distinguishes the following groups:

(d) *Formal analysis based on exact equations*

Because of the great difficulties the only convincing results of rigorous mathematical analysis relate to isotropic turbulence. Dunn and Reid [18] e.g. considered isotropic turbulence in the final period of decay, where the Reynolds and Péclet number are very small, and triple correlations become negligible. So an exact analysis is possible which finally gives

$$Pr_t^{-1} = 2.05\frac{Pr}{1-Pr}\left\{1-\left[\frac{2Pr}{1+Pr}\right]^{3/2}\right\}, \qquad (4.9)$$

for small Re and Pe values.

(e) *Diffusivity models*

The effective diffusivity for heat is expressed in terms of Langrangian length scales and the autocorrelation coefficient of lateral velocity fluctuations. Usually the assumption of homogeneous turbulence is incorporated. A typical result of this group is given by Reynolds [8]

$$Pr_t = \frac{1+C_1Pe^{-1/2}}{1+C_2Re^{-1/2}}, \qquad (4.10)$$

with $C_1 = 86$ and $C_2 = 200$ as empirical constants.

(f) *Wall-layer models*

This group differs from the others in being defined by a field of application. Some models use "renewal-penetration" arguments for the sublayer, others are developed from van Driest's damped mixing-length concept.

(g) *Empirical formulae*

From measurements in boundary layer air flows Rotta [19] found

$$Pr_t = 0.9-0.4(y/\delta)^2, \quad \text{for } Pr = 0.7. \qquad (4.11)$$

Here δ is the boundary layer thickness. The trend, that Pr_t decreases with the wall distance from the value 0.9 at the wall to 0.5 near the outer edge of the boundary layer is in opposition to measurements of Quarmby and Quirk [20] for pipe flow.

Gräber [21] took no account of the wall distance; he only considered the influence of the molecular Prandtl number and proposed

$$Pr_t^{-1} = 0.91+0.13Pr^{0.545},$$

applicable for $0.7 < Pr < 100$.

It should be remembered that only a few methods discussed by Reynolds have been reported here. This has been done in order to make some concluding remarks about first order closure and related assumptions.

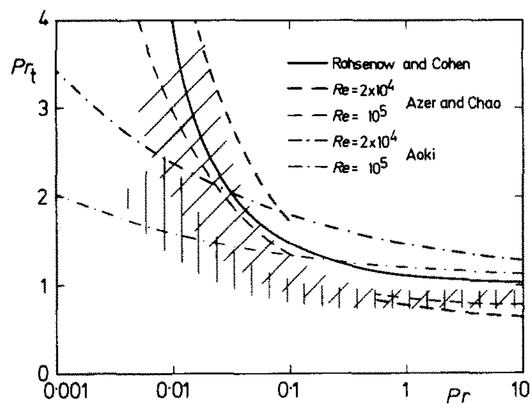

FIG. 2. Some relationships for $Pr_t(Pr, Re)$ compared with experimental results.

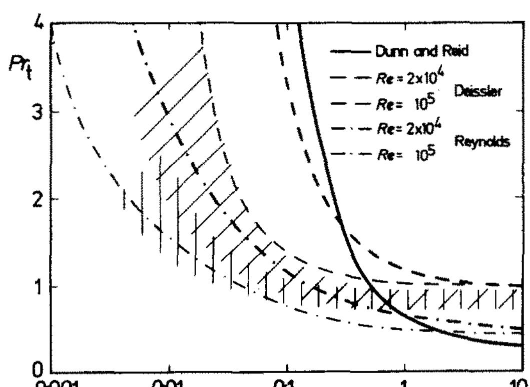

FIG. 3. Some relationships for $Pr_t(Pr, Re)$ compared with experimental results.

In the opinion of the authors they can be roughly divided into two groups:

models which are quite good for practical purposes; they tell us little about the turbulence mechanism (a,b,c,f,g);
models which tell us more about the turbulence mechanism; they are of limited importance for practical purposes (d,e).

Figures 2 and 3 show some of the discussed formulae compared with the experimental information given in Fig. 1. Evidently no relationship gives good agreement with the measurements in the whole range of the molecular Prandtl number.

5. PROPOSAL FOR A SECOND ORDER CLOSURE ASSUMPTION

The aim of our work is to derive:

an expression for the turbulent Prandtl number based on modeled transport equations for the turbulent kinetic energy as well as the Reynolds heat flux; and
an expression for the turbulent Schmidt number based on modeled transport equations for the turbulent kinetic energy and the Reynolds mass flux.

In order to do this we follow the original work of Prandtl [22] who modeled the transport equation for the turbulent kinetic energy:

$$k = \tfrac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}). \tag{5.1}$$

For thin shear flow layers this transport equation reads, see Rotta [19] for example:

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} + \overline{u'v'}\frac{\partial u}{\partial y} + \varepsilon + \frac{\partial}{\partial y}\left|\overline{\left(k + \frac{p'}{\rho}\right)v'}\right| = 0. \tag{5.2}$$

Here ε is the turbulent dissipation (v_j = velocity vector, x_k = space co-ordinate vector):

$$\varepsilon = \nu\,\overline{\frac{\partial v_j'}{\partial x_k}\left(\frac{\partial v_j'}{\partial x_k} + \frac{\partial v_k'}{\partial x_j}\right)}. \tag{5.3}$$

Following Prandtl we use his original modeling assumptions:

Turbulent shear stress

$$-\overline{u'v'} = \varepsilon_m\frac{\partial u}{\partial y} = C_\tau\sqrt{k}L\frac{\partial u}{\partial y}; \tag{5.4}$$

"production"

$$-\overline{u'v'}\frac{\partial u}{\partial y} = C_\tau\sqrt{k}L\left(\frac{\partial u}{\partial y}\right)^2; \tag{5.5}$$

"dissipation"

$$\varepsilon = C_D\frac{k^{3/2}}{L}; \tag{5.6}$$

"diffusion"

$$-\frac{\partial}{\partial y}[\ldots\ldots] = C\frac{\partial}{\partial y}\left|\sqrt{k}L\frac{\partial k}{\partial y}\right|. \tag{5.7}$$

L is a turbulence length scale. In the following, only equation (5.4) for the Reynolds sheat stress is used. The modeling assumptions (5.5—5.7) are given to lead over to a similar modeling of the transport equation for the Reynolds heat flux.

To get the transport equation for $\overline{T'v_j'}$ one multiplies the equation for the instantaneous value of temperature $(T+T')$ with the velocity fluctuation v_j', and adds it to the x_j-component of the Navier–Stokes equations multiplied by the temperature fluctuation T'. Upon temperal averaging the result reads:

$$v_k\frac{\partial\overline{T'v_j'}}{\partial x_k} = \underbrace{-\overline{v_j'v_k'}\frac{\partial T}{\partial x_k} - \overline{v_k'T'}\frac{\partial v_j}{\partial x_k} + \frac{1}{\rho c_p}\overline{v_j'v_k'}\frac{\partial p}{\partial x_k}}_{\text{"Production"}}$$
$$\underbrace{+\frac{1}{\rho c_p}\left|\rho\overline{\varepsilon' v_j'} + \lambda\overline{v_j'\frac{\partial^2 T'}{\partial x_k^2}} + c_p\eta\overline{T'\frac{\partial^2 v_j'}{\partial x_k^2}}\right|}_{\text{"Dissipative terms"}}$$
$$\underbrace{+\frac{1}{\rho c_p}\left|-c_p\overline{T'\frac{\partial p'}{\partial x_j}} + \overline{v_j'\frac{\partial p'}{\partial t}} + v_k\overline{v_j'\frac{\partial p'}{\partial x_k}} + \overline{v_j'v_k'\frac{\partial p'}{\partial x_k}}\right|}_{\text{"Redistributive terms"}}\underbrace{-\frac{\partial}{\partial x_k}(\overline{T'v_j'v_k'})}_{\text{"Diffusion"}}$$

In order to get analogy to the transport equations for the shear stress $\overline{u'v'}$ and for the turbulent kinetic energy k, the "dissipative" and "redistributive" terms have to be transformed. For example the first "redistributive" term is transformed by:

$$-\frac{1}{\rho}\overline{T'\frac{\partial p'}{\partial x_j}} = \underbrace{-\frac{1}{\rho}\frac{\partial}{\partial x_j}\overline{p'T'}}_{\text{"Diffusion"}} + \underbrace{\frac{1}{\rho}\overline{p'\frac{\partial T'}{\partial x_j}}}_{\text{"Redistribution"}}$$

Introducing the molecular Prandtl number the "dissipative" terms can be transformed as follows:

$$\frac{\lambda}{\lambda c_p}\overline{v'_j\frac{\partial^2 T'}{\partial x_k^2}}+\frac{\eta}{\rho}\overline{T'\frac{\partial^2 v'_j}{\partial x_k^2}}=\nu\left[\frac{1}{Pr}\overline{v'_j\frac{\partial^2 T'}{\partial x_k^2}}+\overline{T'\frac{\partial^2 v'_j}{\partial x_k^2}}\right]$$

$$=\nu\Bigg[\underbrace{\frac{\partial}{\partial x_k}\left(\frac{1}{Pr}\frac{\partial}{\partial x_k}\overline{T'v'_j}+\left(1-\frac{1}{Pr}\right)\overline{T'\frac{\partial v'_j}{\partial x_k}}\right)}_{\text{"Diffusive"}}-\underbrace{\frac{Pr+1}{Pr}\overline{\frac{\partial T'}{\partial x_k}\frac{\partial v'_j}{\partial x_k}}}_{\text{"Dissipative"}}\Bigg]$$

In the case of $Pr = 1$ this is analogous to the corresponding terms in the transport equation for $\overline{u'v'}$:

$$\nu\left[\overline{v'_j\frac{\partial^2 T'}{\partial x_k^2}}+\overline{T'\frac{\partial^2 v'_j}{\partial x_k^2}}\right]=\nu\left[\frac{\partial^2 \overline{T'v'_j}}{\partial x_k^2}-2\,\overline{\frac{\partial T'}{\partial x_k}\frac{\partial v'_j}{\partial x_k}}\right].$$

The transport equation for $\overline{T'v'_j}$ now reads for arbitrary Prandtl numbers:

$$v_k\frac{\partial \overline{T'v'_j}}{\partial x_k}=-\overline{v'_jv'_k}\frac{\partial T}{\partial x_k}-\overline{v'_kT'}\frac{\partial v_j}{\partial x_k}+\frac{1}{\rho c_p}\overline{v'_jv'_k}\frac{\partial p}{\partial x_k}+\frac{1}{c_p}\overline{\varepsilon' v'_j}-\nu\frac{Pr+1}{Pr}\overline{\frac{\partial T'}{\partial x_k}\frac{\partial v'_j}{\partial x_k}}$$
$$-\frac{1}{\rho c_p}\overline{p'\left[\frac{\partial v'_j}{\partial t}+v'_k\frac{\partial v'_j}{\partial x_k}+v_k\frac{\partial v'_j}{\partial x_k}+c_p\frac{\partial T'}{\partial x_j}\right]}$$
$$-\frac{\partial}{\partial x_k}\left[\overline{T'v'_jv'_k}-\frac{\nu}{Pr}\frac{\partial \overline{T'v'_j}}{\partial x_k}-\nu\frac{Pr-1}{Pr}\overline{T'\frac{\partial v'_j}{x_k}}\right.$$
$$\left.-\frac{1}{\rho c_p}v_k\overline{p'v'_j}+\frac{1}{\rho}\overline{p'T'}\delta_{jk}-\overline{v'_kv'_jp'}\frac{1}{\rho c_p}\right].$$

For thin shear flow layers it follows, see Jischa [23] for example:

$$\underbrace{u\frac{\partial}{\partial x}(\overline{T'v'})+v\frac{\partial}{\partial y}(\overline{T'v'})}_{\text{Convection}}+\underbrace{\overline{v'^2}\frac{\partial T}{\partial y}-\frac{\overline{u'v'}}{\rho c_p}\frac{\mathrm{d}p}{\mathrm{d}x}}_{\text{"Production"}}$$

$$\underbrace{-\frac{1}{c_p}\overline{v'\varepsilon}+\nu\frac{Pr+1}{Pr}\overline{\frac{\partial v'}{\partial x_j}\frac{\partial T'}{\partial x_j}}}_{\text{"Dissipation"}}\underbrace{-\overline{\frac{p'}{\rho}\frac{\partial T'}{\partial y}}+\ldots}_{\text{"Redistribution"}}+\underbrace{\frac{\partial}{\partial y}[\overline{T'v'^2}-\ldots]}_{\text{"Diffusion"}}=0. \quad (5.8)$$

The "redistribution" and "diffusion" terms are not given in detail here; the quotation marks should refer to the fact that the names of the terms are problematic.

The following discussion is now made for fully developed duct flow and therefore zero convective terms in equation (5.8).

In analogy to the modeling above, equation (5.5), neglecting the pressure gradient production, we choose for the production term:

$$\overline{v'^2}\frac{\partial T}{\partial y}=ak\frac{\partial T}{\partial y}. \quad (5.9)$$

Now obviously the simplest case is the assumption production = dissipation (or redistribution) neglecting the diffusion term. Usually all "dissipation" terms in the transport equations as well as all terms containing the molecular viscosity ν are neglected except the turbulent dissipation ε. The arguments are based on the fact that the dissipative correlation in equation (5.8) is zero in isotropic turbulence and will be negligible also in non-isotropic turbulence, provided that the turbulence Reynolds number is high. This means that the redistribution terms limit the growth of the Reynolds heat flux. Therefore the redistribution terms play a similar role to the dissipation terms and no difference between them should be made for modeling assumptions. So the authors suggest:

$$\text{Dissipation and/or redistribution} = K_D\frac{\sqrt{k}}{L}\overline{T'v'}, \quad (5.10)$$

in analogy to equation (5.6). The assumption production = dissipation (or redistribution) leads to:

$$ak\frac{\mathrm{d}T}{\mathrm{d}y}+K_D\frac{\sqrt{k}}{L}\overline{T'v'}=0, \quad (5.11)$$

that means

$$-\overline{T'v'}=\frac{a}{K_D}\sqrt{k}L\frac{\mathrm{d}T}{\mathrm{d}y}, \quad (5.12)$$

and the turbulent Prandtl number follows together with equation (5.4)

$$Pr_t = \frac{C\,K_D}{a} = \text{constant}, \qquad (5.13)$$

which corresponds to the results of simple mixing length concepts which give $Pr_t = 1$ or 0.5.

In the next step the diffusion term in equation (5.8) shall be taken into account. This has been done by the authors in a preceding paper [24]. For fully developed turbulent channel flow this leads to the result:

$$Pr_t = \frac{A}{B+f(Re, y/H)} = Pr_t(Re, y/H), \qquad (5.14)$$

for $Pr = 1$. Numerical calculations (a set of ordinary differential equations has to be solved) gave the result shown in Fig. 4.

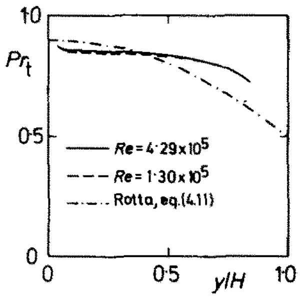

FIG. 4. $Pr_t(Re, y/H)$ from [24] for $Pr = 1$ and channel flow.

The numerical results show for $Pr = 1$ a very weak influence of the Reynolds number which is in agreement with experimental results (see Fig. 1). Furthermore the dependence of the wall distance is comparable with that given by Rotta, equation (4.11).

Figure 4 shows, together with experimental information, that the assumption of a negligible influence of the wall distance is not too bad. That means that production and dissipation (or redistribution) are the dominant terms in equation (5.8). So we now neglect the diffusion term again but we focus our attention on the fact, that the second dissipation term in equation (5.8) depends on the molecular Prandtl number. The modeling given in equation (5.10) should profit by this dependence in the following way:

Dissipation and/or redistribution

$$= \left(K_D + K_p \frac{Pr+1}{Pr}\right)\frac{\sqrt{k}}{L}\overline{T'v'}. \qquad (5.15)$$

Instead of equation (5.11) we now have:

$$ak\frac{\mathrm{d}T}{\mathrm{d}y} + \left(K_D + K_p \frac{Pr+1}{Pr}\right)\frac{\sqrt{k}}{L}\overline{T'v'} = 0, \qquad (5.16)$$

and the turbulent Prandtl number finally reads

$$Pr_t(Pr) = \frac{C_t}{a}\left(K_D + K_p\frac{Pr+1}{Pr}\right)$$

$$= A + B\frac{Pr+1}{Pr} = C + \frac{B}{Pr}. \qquad (5.17)$$

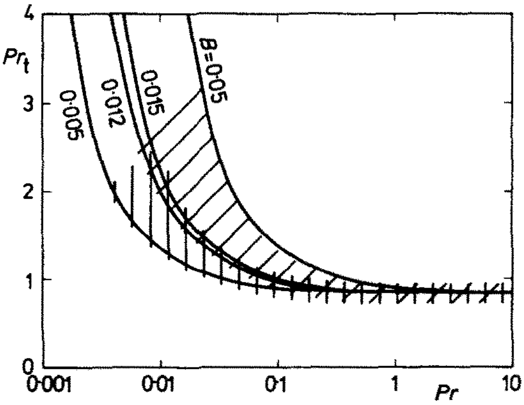

FIG. 5. $Pr_t(Pr, Re)$ from equation (5.17).

This formula is extremely simple, but it corresponds very well with experimental data as Fig. 5 shows. The two constants C and B have to be fitted on these data. We have chosen

$$C = 0.85, \text{ that means } Pr_t = 0.85 \text{ for } Pr \gg 1.$$

The coefficient B seems to be a function of the Reynolds number, which we have not considered because we have neglected the diffusion term in equation (5.8). Using the experimental information of Fig. 1 which is shown in Fig. 5 again we can improve the agreement with experimental data by assuming $B = B(Re)$. The considered experimental data suggest approximately:

$$B = 0.012 \text{ to } 0.05, \quad \text{for } Re = 2\times 10^4$$

$$B = 0.005 \text{ to } 0.015, \quad \text{for } Re = 10^5.$$

Analogously we shall model the transport equation for the Reynolds mass flux $\overline{c'v'}$ which reads for thin shear flow layers, see Jischa [23] for example:

$$\underbrace{u\frac{\partial}{\partial x}(\overline{c'v'}) + v\frac{\partial}{\partial y}(\overline{c'v'})}_{\text{Convection}} + \underbrace{\overline{v'^2}\frac{\partial c}{\partial y}}_{\text{"Production"}}$$

$$+ \underbrace{v\left(\frac{Sc+1}{Sc}\right)\overline{\frac{\partial v'}{\partial x_j}\frac{\partial c'}{\partial x_j}}}_{\text{"Dissipation"}} - \underbrace{\overline{\frac{p'}{\rho}\frac{\partial c'}{\partial y}}}_{\text{"Redistribution"}}$$

$$+ \underbrace{\frac{\partial}{\partial y}[\overline{c'v'^2} - \ldots]}_{\text{"Diffusion"}} = 0. \qquad (5.18)$$

The "diffusion" term is not given in detail because we shall neglect it in the following discussion. We consider again the case of fully developed duct flow with zero convective terms in equation (5.18). In analogy to equation (5.9) we model the production term:

$$\overline{v'^2}\frac{\partial c}{\partial y} = ak\frac{\partial c}{\partial y}. \qquad (5.19)$$

In order to get the same result for Sc_t as for Pr_t we suggest

$$\text{redistribution} = C_R\frac{\sqrt{k}}{L}\overline{c'v'}, \qquad (5.20)$$

and

$$\text{dissipation} = C_S \frac{Sc+1}{Sc} \frac{\sqrt{k}}{L} \overline{c'v'}, \quad (5.21)$$

according to equation (5.15).

The modeled transport equation for $\overline{c'v'}$ reads now:

$$ak \frac{\partial c}{\partial y} + \left(C_R + C_S \frac{Sc+1}{Sc} \right) \frac{\sqrt{k}}{L} \overline{c'v'} = 0. \quad (5.22)$$

The "dissipation" term in the transport equation for $\overline{c'v'}$ depends on the molecular Schmidt number in the same way as the second "dissipation" term in the transport equation for $\overline{T'v'}$ depends on the molecular Prandtl number.

Using equation (5.4) for ε_m the turbulent Schmidt number is given by the relationship:

$$Sc_t(Sc) = \frac{c}{a} \left(C_R + C_S \frac{Sc+1}{Sc} \right)$$

$$= D + E \frac{Sc+1}{Sc} = F + \frac{E}{Sc}. \quad (5.23)$$

Assuming Sc_t to be the same function of Sc as $Pr_t(Pr)$, we get $F = C$ and $E = B$. Then Fig. 5 can be used too for the relationship $Sc_t(Sc, Re)$.

6. CONCLUDING REMARKS

Obviously equation (5.17) agrees better with experimental data than the formulae shown in Figs. 2 and 3. Maybe one reason is the fact, that the influence of the molecular Prandtl number by the factor $(Pr+1)/Pr$ in equation (5.17) is no result of a modeling assumption because it appears in the transport equation (5.8) for the Reynolds heat flux.

Of course it is a disadvantage of equation (5.17) that it does not express Pr_t as a function of the wall distance and that the influence of the Reynolds number was taken into consideration by adaptation of experimental data. But in our opinion no simple algebraic formula like equation (5.17) or others can do this except by experimental adaptation.

In order to get information about $Pr_t = f(Pr, Re, y)$ and $Sc_t = f(Sc, Re, y)$ the whole flow, temperature and concentration field has to be considered. That means the governing boundary layer and transport equations have to be solved numerically. It is the aim of the authors to combine the considerations presented here with the briefly reported work [24] in order to get information about the relationships $Pr_t(Pr, Re, y)$ and $Sc_t(Sc, Re, y)$.

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SUR L'ESTIMATION DES NOMBRES DE PRANDTL ET DE SCHMIDT TURBULENTS A PARTIR DES EQUATIONS DE TRANSPORT

Résumé—On présente des formules pour les nombres de Prandtl et de Schmidt turbulents. Elles sont basées sur les équations de transport de l'énergie cinétique de turbulence, du flux de chaleur turbulent et du flux massique turbulent. Les résultats sont des formules simples, $Pr_t = C+B/Pr$ et $Sc_t = F+E/Sc$, où les constantes ont été ajustées à partir des résultats expérimentaux. Ceci a été fait pour les constantes dans l'expression du nombre de Prandtl turbulent qui est en très bon accord avec l'expérience. L'accord peut être amélioré en supposant B fonction du nombre de Reynolds. Le résultat est spécialement intéressant pour les métaux liquides car le nombre de Prandtl turbulent dépend fortement du faible nombre de Prandtl moléculaire. L'influence du nombre de Reynolds, qui augmente quand le nombre de Prandtl diminue, semble être faible.

ÜBER DIE VORHERSAGE TURBULENTER PRANDTL- UND SCHMIDT-ZAHLEN AUS MODELLIERTEN TRANSPORTGLEICHUNGEN

Zusammenfassung—Für die turbulente Prandtl- und Schmidt-Zahl werden Beziehungen vorgestellt, die auf modellierten Transportgleichungen für die Turbulenzenergie, für den turbulenten Wärmestrom und für den turbulenten Massendiffusionsstrom beruhen. Als Ergebnis erhält man einfache Zusammenhänge $Pr_t = C+B/Pr$ und $Sc_t = F+E/Sc$, wobei die Konstanten an experimentelle Daten angepaßt werden müssen. Dies wurde durchgeführt für die Konstanten in der Beziehung für die turbulente Prandtl-Zahl; die Übereinstimmung mit Experimenten ist sehr gut. Setzt man B als eine Funktion der Reynolds-Zahl an, kann die Übereinstimmung verbessert werden. Das Ergebnis ist besonders interessant für flüssige Metalle, da die turbulente Prandtl-Zahl im Bereich kleiner Prandtl-Zahlen offenbar sehr stark von der molekularen Prandtl-Zahl abhängt. Der Einfluß der Reynolds-Zahl, der mit abnehmender Prandtl-Zahl zunimmt, scheint schwächer zu sein.

О РАСЧЁТЕ ТУРБУЛЕНТНЫХ ЧИСЕЛ ПРАНДТЛЯ И ШМИДТА С ПОМОЩЬЮ МОДЕЛЬНЫХ УРАВНЕНИЙ ПЕРЕНОСА

Аннотация — Представлены соотношения для турбулентных чисел Прандтля и Шмидта, которые базируются на модельных уравнениях переноса для турбулентной кинетической энергии, турбулентного теплового потока и турбулентного потока массы. Получены простые формулы $Pr_t = C + B/Pr$ и $Sc_t = F + E/Sc$ с константами, определяемыми экспериментально. Найдены значения констант в соотношении для турбулентного числа Прандтля, которое очень хорошо согласуется с экспериментом. Это согласие может быть еще более улучшено, если предположить, что B является функцией числа Рейнольдса. Результат представляет особый интерес для жидких металлов, так как в этом случае турбулентное число Прандтля сильно зависит от Pr.